Atmospheric contribution to the dissipation of the gravitational tide of Phobos on Mars

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1 Here, we investigate the possibility of a significant atmospheric contribution to the tidal dissipation of the Phobos-Mars system. We apply the classical tidal theory and find that most of the gravitational forcing is projected onto the first symmetric Hough mode which has an equivalent depth of about 57 km and is significantly trapped in the vertical. Therefore, no significant dissipation occurs through the vertical propagation of energy and subsequent breaking of the tidal wave as the wave amplifies with height. Alternatively, from the energy stored in the first trapped mode we estimate that the time scale required for the dissipative mechanisms to account for the total dissipation of the tides is of order 105 s. This time scale is unrealistically short, since it would contradict observations of propagating thermal tides in Mars atmosphere. Therefore we conclude that the dissipation of the tidal potential that explains the observed acceleration of Phobos most likely occurs within the solid planet. Citation: Rondanelli, R., V. Thayalan, R. S. Lindzen, and M. T. Zuber (2006), Atmospheric contribution to the dissipation of the gravitational tide of Phobos on Mars, Geophys. Res. Lett., 33, L15201, doi:10.1029/2006GL026222.

2. Summary of the Theory

2.1. Classical Tidal Theory

3 Here, we briefly summarize the assumptions and main results of the application of classical tidal theory to a thin atmospheric shell as assumed for Mars. Detailed derivations and discussions can be found in work by Chapman and Lindzen [1970]. The tidal fields are assumed to be small perturbations about a basic state, so for instance the total density field \( \rho_{\text{tot}} \) can be written as,

\[
\rho_{\text{tot}} = \rho_0(z) + \rho(0, \phi, z, t),
\]

where \( \rho_0 \) is the basic state density which depends only on \( z \) the height above the surface, and \( \rho \) is the perturbation. The coordinates are the colatitude \( \theta \) measured from the axis of rotation, the azimuth \( \phi \), the height \( z \) from the surface of the planet and the time \( t \). The basic state is in hydrostatic balance and can be considered at rest with respect to the phase speed of the tide \( c \sim 240 \text{ m/s} > U \sim 10 \text{ m/s} \), where \( c \) is the dimensional phase speed of the tide and \( U \) is a typical scale for the horizontal velocity in Mars’ atmosphere). The inviscid momentum equations, the adiabatic thermodynamic equation and the ideal gas law are linearized around this basic state.

4 Since we are interested in the steady state response to a forcing with a known period and magnitude, the resultant tidal fields are assumed to be periodic in time, having the same frequency \( \sigma \) as the forcing (\( \sigma = 2 \omega_L \)), where \( \omega_L = \omega - \omega_M \) is the relative rotation of the moon with respect to the planet, \( \omega \) is the rotation rate of the planet and \( \omega_M \) is the rotation rate of the satellite around the planet (Table 1). Here, we are interested in the response to the dominant semi-diurnal component of the forcing having a wave number \( s = 2 \). Under these assumptions the equations governing the atmospheric response are separable into an equation that contains only vertical dependence and an equation that contains only latitudinal dependence (time and azimuthal dependence are given by the forcing) [Chapman and Lindzen, 1970]. The latitudinal dependence of the perturbation fields is governed by the Laplace Tidal equation,

\[
\frac{d}{d\mu} \left( \frac{1 - \mu^2}{f^2 - \mu^2} \frac{d\Theta^\mu_n}{d\mu} \right) - \frac{1}{f^2 - \mu^2} \left( \frac{s f^2 + \mu^2}{f^2 - \mu^2} + \frac{s^2}{1 - \mu^2} \right) \Theta^\sigma_n + \frac{4a^2 \omega^2}{gh^2} \Theta^\sigma_n = 0,
\]

where \( \mu = \cos \theta \), \( f = \sigma/2\omega \), \( g \) is the gravitational acceleration on Mars, and \( a \) is the radius of Mars. \( \Theta^\mu_n \) are the solutions of the equation, known as Hough functions and \( h^\mu_n \) are the equivalent depths which represent the eigenvalues of the problem, the subscript \( n \) is used to identify the eigenfunc-
Table 1. Parameters for the Calculation of the Forcing

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Parameter Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega )</td>
<td>Rotation rate of the planet</td>
<td>( 7.08 \times 10^{-3} ) rad/s</td>
</tr>
<tr>
<td>( \omega_M )</td>
<td>Rotation rate of the satellite</td>
<td>( 2.27 \times 10^{-4} ) rad/s</td>
</tr>
<tr>
<td>( \omega_L )</td>
<td>Relative rotation rate</td>
<td>(-1.57 \times 10^{-4} ) rad/s</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>Frequency of the forcing</td>
<td>(-3.14 \times 10^{-4} ) rad/s</td>
</tr>
<tr>
<td>( M_p )</td>
<td>Mass of Phobos</td>
<td>( 1.08 \times 10^{25} ) kg</td>
</tr>
<tr>
<td>( D )</td>
<td>Distance between Phobos and Mars</td>
<td>( 9.38 \times 10^{4} ) km</td>
</tr>
<tr>
<td>( a )</td>
<td>Radius of Mars</td>
<td>( 3.40 \times 10^{6} ) m</td>
</tr>
<tr>
<td>( \epsilon )</td>
<td>Inclination of the orbit of Phobos</td>
<td>1°</td>
</tr>
</tbody>
</table>

The vertical structure of the modes is found through the solution of the equation

\[
\frac{d^2 y_n}{dx^2} + Q^2 y_n = 0, \tag{3}
\]

where \( Q^2 = (N^2 H^2/gh_n - 1/4) \) is the index of refraction squared, \( N \) is the Brunt-Väisälä frequency, and \( H \) is the scale height of the basic state (\( H = RT_0/\rho \), where \( R \) is the gas constant for the Mars atmosphere and \( T_0 \) is the basic state temperature). The functions \( y_n \) are related to the rest of the tidal fields through the three dimensional divergence perturbation \( \chi_n \) as \( y_n = \chi_n e^{x/2} \), where \( x = \int dz/H \) is the log-pressure vertical coordinate. Assuming a perfectly spherical planet, the lower boundary condition is \( w_{\text{in}} = w = 0 \) and can be written as,

\[
\gamma h_n \left[ \frac{dy_n}{dx} + \left( \frac{H}{\rho_n} - \frac{1}{2} \right) y_n \right]_{x=0} = \frac{i \Omega_n}{g}, \tag{4}
\]

where \( \gamma \) is the ratio of the heat capacity at constant pressure and the heat capacity at constant volume and \( i \) is the complex imaginary unit. \( \Omega_n \) refers to the component of the gravitational forcing projected onto the Hough function of order \( n \).

At the upper boundary, we impose the radiation condition, which for propagating modes is equivalent to choosing the solution that has an upward flux of energy [Wilkes, 1949]. Once the gravitational forcing and the eigenvalues are known, equation 3 can be numerically integrated following a procedure similar to the one described by Lindzen [1990, p. 297]. In sum, any given perturbation field, for instance the density, can be written as,

\[
\rho(0, \phi, z, t) = \sum_n \rho_n(\phi) \Theta_n^\phi(\theta) e^{i(\sigma t + \phi)}, \tag{5}
\]

For simplicity, we have assumed a perfectly spherical Mars, without including any energy dissipation in the form of frictional drag with the surface topography, despite the degree 2,2 equatorial features that includes the Tharsis bulge [Smith et al., 1999].

2.2. Tidal Potential

The gravitational tidal potential exerted by Phobos at an arbitrary point \( \mathbf{a} \) over the surface of the planet, measured from the center of Mars can be written as

\[
\Omega = -\frac{GM_p}{|\mathbf{D} - \mathbf{a}|} = -\frac{GM_p}{D} \sum_{n=0}^{\infty} \left( \frac{a}{D} \right)^n P_n(\mu), \tag{6}
\]

where \( G \) is the gravitational constant, \( M_p \) the mass of the satellite, and \( \mathbf{D} \) is the position vector of the satellite in a coordinate system centered at the center of mass of the planet (\( D = |\mathbf{D}| \)). The first two terms in the series expansion 6 are a constant potential with no consequence in the forcing, and a term which gives an homogeneous forcing equivalent to the acceleration experienced by the center of mass of the planet. For the purpose of specifying the tidal forcing we will be concerned only with the term of order \( \left( \frac{a}{D} \right)^2 \). Then, the semidiurnal component of the gravitational potential due to satellite of mass \( M_p \) has the form,

\[
\Omega \approx -\sqrt{\frac{3GM_p a^2}{5D^3}} P_2(\cos 0) \left( 1 - \frac{\mu^2}{2} \right), \tag{7}
\]

where \( P_2(\mu) \) is the normalized associated Legendre polynomial of degree 2, and \( \mu = \sin \theta \) is the orbital inclination of the satellite. Despite Phobos being relatively close to the surface of Mars compared to the Moon, the smaller mass of Phobos dictates that the gravitational potential of the Moon over the Earth is still about two orders of magnitude larger than the gravitational potential of Phobos over Mars.

3. Results

3.1. Characteristics of the Response

As expected from the relatively large value of \( f \), the expansion of the \( \Theta_n^\phi \) are dominated by the contribution of the corresponding \( P_2^\phi \), that is, only a small correction from the spherical harmonics appears as a result of the rotation of the planet relative to the rotation of the moon. As a consequence, since the gravitational forcing can be expressed in terms of \( P_2^\phi \) most of the excitation goes into the first symmetric Hough mode, \( \Theta_2^\phi \). From Table 2 we see that the contribution of each of the higher modes to the expansion of the gravitational potential is reduced by about two orders of magnitude relative to the preceding mode.

One can get a rough idea of the vertical propagation of the modes by considering an isothermal version of equation 3. In that case the quantity \( \frac{-\frac{\kappa}{H}}{\frac{\kappa}{H}} \) becomes simply \( \frac{\kappa^2}{H^2} \), where \( \kappa = (\gamma - 1)/\gamma \). Therefore, when \( 4 \frac{\kappa}{H} h_n \leq 1 \), the corresponding mode propagates in the vertical, whereas when it is more than one the mode is vertically trapped. From Table 2 we see that at least the first two modes, which receive the bulk of the gravitational forcing, are vertically trapped. The consequence for the problem at hand is that the evaluation of the tidal dissipation cannot be done directly by calculating the upward energy flux \( \varphi_W \),
where \( p \) and \( w \) are the pressure and vertical velocity tidal fields, respectively and the bar refers to average over one wavelength. We know that for vertically propagating modes the tidal fields grow with height as \( r_0^2/C_0^2 \) and therefore it is expected that at some height the waves will become unstable and will break, dissipating the upward energy flux through the generation of turbulence. Such an explicit calculation can still be done for the higher modes which propagate in the vertical, that is \( n/C_2 \neq 6 \), however, as we will see, we find very modest contributions through this mechanism.

\[ P_2,2 = 0.99982 \]
\[ P_4,2 = -0.01871 \]
\[ P_6,2 = 0.00016 \]
\[ P_8,2 = -0.00000 \]

Also shown are the equivalent depths corresponding to each mode (exact and approximated for large \( f \)) and the parameter \( 4 kH/h_0 \), indicative of the vertical propagation of each mode in an isothermal atmosphere \((T = 200 \text{ K})\). The data is for Mars and Phobos with \( f = -2.2152 \) and \( s = 2 \).

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A more realistic approach is to consider a varying vertical profile of temperature and evaluate numerically equation 3. We use as a reference the vertical profiles taken during the descent of the Viking landers [Seiff and Kirk, 1977]. It is evident from the Viking observations that these profiles show not only the background state of the atmosphere but also superimposed wave activity. We construct two profiles V1 and V2 (see Figure 1a) by smoothing the original Viking soundings and taking only the mean vertical lapse rate over five or six layers of about 40 km in height, roughly resembling the observed profiles. We use 1000 vertical levels over a vertical domain of 200 km using the numerical equivalents of the boundary conditions. Note that the magnitude of the tidal fields is not particularly sensitive to the characteristics of the basic temperature profile.

We present results for the first mode \((n = 2)\). We see in Figure 1b that the refractive index \( Q^2 \) remains negative over the entire atmosphere. As a consequence the magnitude of the velocity and temperature tidal fields grow only slowly with height, the slow growth is due to the competing effect of the decrease in density with height which amplifies the response and the decaying nature of the solution. Consequently these tidal fields remain within the same order of magnitude throughout the atmosphere (Figure 1c). Our model does not consider explicitly any dissipative mechanism that in the real atmosphere will likely damp any slow growth with height. The behavior of the pressure perturbation shows a strong decay in amplitude with height, which illustrates the fact that the tidal forcing is concentrated near the ground and, for this particular mode, the response does not propagate significantly with height (Figure 1d).

### Table 2. Coefficients of the Expansion of the First Four Hough Modes in Terms of Normalized Associated Legendre Polynomials

<table>
<thead>
<tr>
<th>( n )</th>
<th>( 2 )</th>
<th>( 4 )</th>
<th>( 6 )</th>
<th>( 8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h_0 ) [km]</td>
<td>57.7028</td>
<td>14.9279</td>
<td>6.82552</td>
<td>3.91756</td>
</tr>
<tr>
<td>( \sigma_{n+1}^{1/2} )</td>
<td>53.3789</td>
<td>15.5534</td>
<td>7.35891</td>
<td>4.2823</td>
</tr>
<tr>
<td>( 4 \kappa H/h_0 )</td>
<td>0.1604</td>
<td>0.6203</td>
<td>1.3566</td>
<td>2.3636</td>
</tr>
</tbody>
</table>

| \( \Theta_2 \) | 0.99982 | 0.01869 | 0.00086 | 0.00005 |
| \( \Theta_4 \) | -0.01871 | 0.99834 | 0.05434 | 0.00352 |
| \( \Theta_6 \) | 0.00016 | -0.05445 | 0.99463 | 0.08770 |
| \( \Theta_8 \) | -0.00000 | 0.00127 | -0.08799 | 0.98892 |

*Also shown are the equivalent depths corresponding to each mode (exact and approximated for large \( f \)) and the parameter \( 4 kH/h_0 \), indicative of the vertical propagation of each mode in an isothermal atmosphere \((T = 200 \text{ K})\). The data is for Mars and Phobos with \( f = -2.2152 \) and \( s = 2 \).*

### Figure 1.

(a) Basic temperature profiles for the two idealized soundings, Viking 1 (solid line) and Viking 2 (dashed line) (Profiles are idealizations from the Viking soundings [Seiff and Kirk, 1977]). (b) \( Q^2 \), index of refraction squared for the first Hough mode. (c) Magnitude of the vertical component of the zonal velocity perturbation \(|v_2(z)|\) for the first Hough mode. (d) Magnitude of the vertical component of pressure perturbation \(|p_2(z)|\) for the first Hough mode.
To illustrate the relative (and small) importance of the higher order terms in the expansions of the tidal response Figures 2 and 3 show the latitudinal variation of the magnitude of the surface perturbation of $u$ and $v$ respectively. It can be seen that the magnitude of the sum of the first four modes is about the same order of magnitude ($\sim 10^{-5}$ m/s) and it has almost the same latitudinal distribution as the first Hough mode.

For the calculations in which we explicitly included the temperature profiles, we find that the first propagating mode is $n = 10$ (fifth symmetric mode). As anticipated, the vertical energy flux for this mode is very small and it only contributes a dissipation of the order of $10^{-11}$ W. In the next section we look for an alternative, namely that the dissipation occurs through an unknown process acting on the response to the first mode.

3.2. Energy Calculations

Here, instead of explicitly calculating the dissipation in the atmosphere as the vertical flux of energy of the wave, we explore the consequences of assuming that the known value of the dissipation required to explain the orbital acceleration of Phobos occurs through the dissipation of the tidal energy exclusively in the atmosphere. To this end we calculate the total energy of the tide per unit volume $E$, which can be expressed as the sum of a kinetic and a available potential contribution [see, e.g., Gill, 1982],

$$E = \frac{1}{2} \rho_0 (u^2 + v^2 + w^2) + \frac{1}{2} \rho_0 g z \frac{\partial}{\partial z} \left( \frac{\rho}{\rho_0} \right)^2. \quad (8)$$

As we deduce from the previous section, the energy is concentrated in the gravest mode, $n = 2$ and the higher order contributions are at least two orders of magnitude smaller than for $n = 2$. For the purpose of estimating an order of magnitude for the total tidal energy, it suffices to consider only the contribution of the first mode. We look at the quantity

$$\tau_{\text{diss}} = \int_V \frac{dE}{dt}, \quad (9)$$

where the integral is taken over the volume of the atmosphere, $V$. When $dE/dt = 3.34 \times 10^6 \, W$, the dissipation consistent with the acceleration of Phobos orbit, then $\tau_{\text{diss}}$ is the time scale required to account for such a dissipation within the atmosphere. The value of $\tau_{\text{diss}}$ for the first mode ranges from $1.01 \times 10^2$ to $2.33 \times 10^2$ s, depending on whether the temperature vertical profile used is $V_2$ or $V_1$. The contribution of the higher order modes to the total energy (or to the total dissipation time scale) is negligible.

4. Concluding Remarks

The dissipation time scale obtained in the previous section is extremely short compared to known dissipative processes in the Martian atmosphere such as the radiative damping time constant in the lower atmosphere ($\sim 10^5$ s) [Zurek et al., 1992]. As discussed by Chapman and Lindzen [1970], a local condition for neglecting the dissipation in the formulation of the classical tidal theory is that $\tau_{\text{diss}} \gg \tau_{\text{damp}}$. For the Phobos-Mars system $\tau_{\text{damp}} = 3.18 \times 10^3 \, s$ and therefore our assumed source of dissipation would not only be important in calculating the tidal response but would be a dominant term in such a way as to effectively damp the tidal response. We can be confident that dissipation time scales as short as that required to explain the tidal dissipation resorting exclusively to the Martian atmosphere are not realistic. Solar diurnal and semidiurnal tides have an even larger forcing period, yet still they have been observed in the surface pressure record, and also the wave structure of the Viking landing profiles has been attributed to vertically propagating thermal tides. No such observations would be possible if the $\tau_{\text{diss}}$ were as small as required to match the Phobos orbital acceleration. Therefore our calculations give confidence that the source of the tidal friction most likely resides in the planetary interior.

Figure 3. Same as Figure 2 but for the zonal velocity.
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References


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